Matrices which Jaroslav Zemánek loved

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Będlewo, September 26, 2017

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Jaroslav Zemánek 1946-2017

Chemnitz, ILAS 1996

Part I - Memories

I met Jaroslav Zemánek for the first time at the end of the 1980s or at beginning of the 1990s on a seminar in Warsaw.

After the seminar all participants have decided to drink coffee and talk about recent conferences.

Just when I had to leave for the railway station to return to Wrocław, Jaroslav Zemánek stood up and said "Why are you going? I would like to discuss matrices".

(...)

Part II
Matrices which Jaroslav Zemánek loved

Notation

trace

$$A = [a_{ij}] \in \mathbb{C}^{m\times n}$$

$$\mathrm{trace}(A) = \sum_j a_{jj}$$

zero-trace matrices

$$\mathcal{Z} = \{A : \mathrm{trace}(A) = 0,\ A \in \mathbb{C}^{n\times n}\}$$

singular value decomposition

$$A = U\Sigma V^H$$

$$\Sigma = \mathrm{diag}(\sigma_j(A)), \qquad U, V \quad \text{unitary}$$

singular values: $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq 0$

c_p norms of Schatten

$$\|A\|_p = \left(\Sigma_j(\sigma_j(A))^p\right)^{1/p} \qquad 1 \leq p \leq \infty$$

- spectral norm, $p = \infty$,

$$\|A\|_\infty = \sigma_1(A)$$

- Frobenius norm, $p = 2$,

$$\|A\|_F = \|A\|_2 = \left(\Sigma_j(\sigma_j(A))^2\right)^{1/2}$$

- nuclear (trace) norm, $p = 1$,

$$\|A\|_1 = \Sigma_j \sigma_j(A)$$

unitarily invariant norm

$$\|UA\| = \|AV\| = \|A\|, \qquad U, V \text{ unitary}$$

$$\sigma(A) = [\sigma_1(A), \ldots, \sigma_t(A)]^T$$

Φ symmetric gauge function

Φ is a norm in $\mathbb{R}^t$, which does not depend on the order and signs of components of vector

Let $\|\cdot\|$ be unitarily invariant norm. Then there is Φ such that

$$\|A\| = \Phi(\sigma(A))$$

Some results of Zemánek

- Characterizations of trace
  Petz and Zemánek 1988

- Strict convexity of singular values
  Aupetit, Makai, Zemánek 1996

Schur-Weyl inequality

$$|\mathrm{trace}(A^k)| \leq \mathrm{trace}(|A|^k)$$

Petz and Zemánek 1988

A linear functional f on $\mathbb{M}_n(\mathbb{C})$ is a nonnegative scalar multiple of the trace if and only if the inequality

$$|f(A^k)| \leq f(|A|^k)$$

holds for all matrices A and some index $k = 1, 2, \ldots$.

Operator algebras

General settings admit conceptual proofs.

Strict convexity of singular values
Aupetit, Makai, Zemánek 1996

If A and B are compact operators on Hilbert space, with singular values satisfying

$$\sigma_j(A) = \sigma_j(B) = \sigma_j\left(\frac{A+B}{2}\right)$$

for all $j = 1, 2, \ldots$, then

$$A = B.$$

Two proofs, geometric and analytic, are given.

Zero-trace matrices

Zemánek has drawn my attention to two results of Kittaneh for c_p-norms of Schatten:

- characterization of the zero-trace matrices,
- approximation by zero-trace matrices.

Problem I

Kittaneh 1991

Let A be a complex matrix of order n and $1 \leq p < \infty$. Then

$$\mathrm{trace}(A) = 0 \quad \text{iff} \quad \|I + zA\|_p \geq n^{1/p} \quad \text{for } z \in \mathbb{C}.$$

It does not hold for $p = \infty$ (spectral norm).

Reformulation of the result of Kittaneh, $1 \leq p < \infty$

A has zero trace iff $\quad \min_{z \in \mathbb{C}} \|I + zA\|_p = \|I\|_p$

Question of Zemánek

Why not for $p = \infty$?

My answer

Because: The subgradient of $\|I\|_p$ is unique for $1 \leq p < \infty$. The subgradient of $\|I\|_p$ is not unique for the spectral norm $(p = \infty)$.

Subgradient of $\|X\|$ - definition

$$\{Y : \mathrm{Re}\ \mathrm{tr}(Y^H X) = \|X\|,\ \|Y\|^* \leq 1\}$$

$|Y\|^*$ dual norm

Characterization of subgradients for unitarily invariant norms Watson 1992, Ziętak 1988 and 1993

Ziętak (1997)
generalization of the result of Kittaneh
(problem I)

Let $\|\cdot\|$ be unitarily invariant norm such that the subgradient of $\|I\|$ is unique.

Then A has zero trace iff $\min_{z\in\mathbb{C}} \|I + zA\| = \|I\|$

Problem II - approximation by zero-trace matrices $\mathcal{Z}$

Kittaneh 1991, $1 \leq p \leq \infty$

$$\min_{X \in \mathcal{Z}} \|A - X\|_p = \frac{\mathrm{tr}(A)}{n^{1/q}}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1$$

Ziętak 1996, unitarily invariant norm

$$\min_{X \in \mathcal{Z}} \|A - X\| = \frac{|\mathrm{tr}(A)|}{\|I\|^*} \equiv \|A - \widetilde{X}\|,$$

$\widetilde{X}$ described, for example,

$$\widetilde{X} = A - \frac{\mathrm{tr}(A)}{n} I$$

$\|\cdot\|^*$ dual norm; $\|\cdot\|_q$ dual norm to $\|\cdot\|_p$

Zemánek has drawn my attention to the result of Maher 1990

Moore-Penrose generalized inverse

$$AA^\dagger A = A, \quad A^\dagger A A^\dagger = A^\dagger,$$
$$(AA^\dagger)^H = AA^\dagger, \quad (A^\dagger A)^H = A^\dagger A,$$

Maher 1990, c_p-norm, $1 < p < \infty$

Let $\mathbb{S} = \{X : AXA = A\}$ be the set of generalized inverse of A and $1 < p < \infty$. Then

$$\|X\|_p < \|A^-\|_p \quad X \neq A^-$$

if and only if $X = A^\dagger$.

Ziętak 1997, unitarily invariant norms

Let $\mathbb{S} = \{X : AXA = A\}$. Then

$$\min_{X \in \mathbb{S}} \|X\| = \|A^\dagger\|$$

Let $X \in \mathbb{S}$. Then the following statements are equivalent

- $X = A^\dagger$
- $\sigma_j(X) \leq \sigma_j(A^-)$ for all j and every A^-
- The property:

$$\|X\|_\Phi < \|A^-\|_\Phi \quad X \neq A^-, \quad \text{if and only if } X = A^\dagger$$

holds for every unitarily invariant norm $\|\cdot\|_\Phi$ with Φ strictly monotonic, in particular for the trace norm.

Zemánek - extension to operators?

Spectral approximation of matrix

Approximation of matrix in spectral norm

$$\min_{X \in \mathcal{M}} ||A - X||,$$

where $\mathcal{M}$ closed convex set in $\mathbb{C}^{m \times n}$

- Spectral approximant to A is not unique in general case.
- We select strict spectral approximant
  which is in some sense the best
  among all spectral approximants and it is unique.

Strict spectral approximation, Ziętak 1995

B is strict spectr. approx. of A if the vector $\sigma(A-B)$ of singular values is minimal with respect to ordinary lexicographic ordering in

$$\{\sigma(A-X) : X \in \mathcal{M}\}.$$

- Strict spectral approximant of A always exists and it is unique.

Remark. Lexicographic ordering:

$$[3,3,2,0] \text{ is bigger then } [3,2,2,2].$$

Conjecture

Let $\mathcal{M}$ be linear subspace of matrices and let (c_p-norms of Schatten)

$$\min_{X \in \mathcal{M}} ||A - X||_p = ||A - X_p||_p.$$

Then

$$\lim_{p \to \infty} X_p = X_\infty \quad \text{strict spectral approx.}$$

Shoudl strict convexity of the singular value sequences, shown by Zemánek in 1996, be applied in the proof?

Zemánek was interested in my conjecture.

He encouraging me to write the paper about not complete proof. The paper appears in Banach Center Publications.

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