Advances in numerical linear algebra:

Celebrating the 60th birthday of Nick Higham

A few words on how Nick influences research

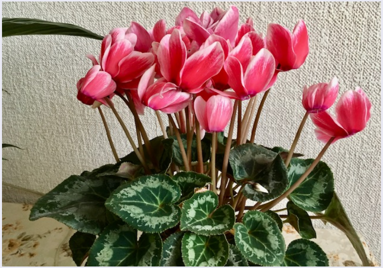
Krystyna Ziętak, Wrocław, Poland

July 8, 2022

Outline

1. Photos and wishes
2. Numerical behavior of Higham's method for polar decomposition
3. Matrix functions

- My congratulations to Nick for so many awards and honours!
- Happy Birthday and many fruitful years!
- I hope all your birthday wishes come true!



my only photo with Nick
Dusseldorf 2006

Alicja Smoktunowicz (Warsaw) sends warm wishes to Nick on the 60 birthday. She also was in Dusseldorf.

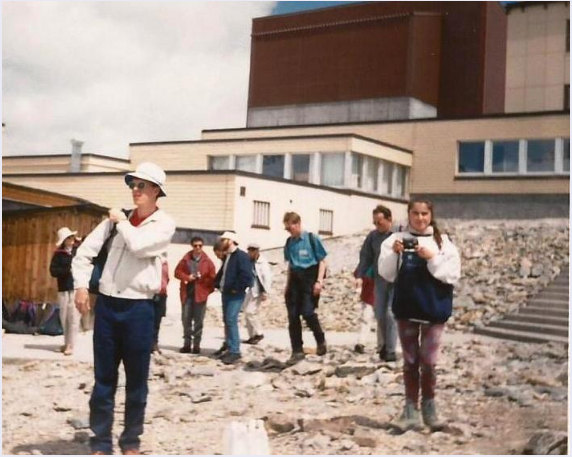
- I met Nick for the first time in 1991 in Manchester.
- I was interested in his method for the polar decomposition.
- The common subject: approximation of matrices.
- I was invited to the Computer Science Department by K.-K. Lau.
- My talk was at a seminar in the Mathematical Department.

Second meeting

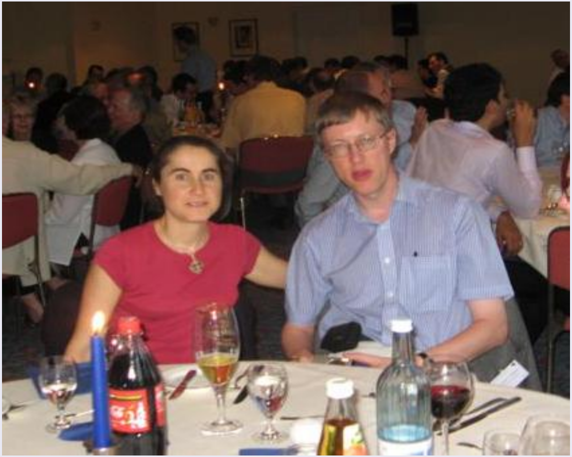
Householder Symposium 1993 in Lake Arrowhead

First meeting with Francoise

Householder Symposium 1996 in Pontresina, Alps



Francoise and Nick, Householder Symposium 2008, Berlin



The last my meeting in Manchester 2013

My 70th birthday

Several important papers of Nick and his coauthors for me

- Computing the Polar Decompositions with Applications. SIAM J. Sci. Statist. Comput. 1986.
- Computing Real Square Roots of a Real Matrix. Linear Algebra Appl. 1987.
- Computing a Nearest Symmetric Positive Semidefinite Matrix. Linear Algebra Appl., 103: 103-118. 1988.
- Matrix Nearness Problems and Applications. In Gover, M. J. C. and Barnett, S. 1989.
- Computing the Polar Decomposition and the Matrix Sign Decomposition in Matrix Groups. SIAM J. Matrix Anal. Appl. 2004.
- A Schur-Newton Method for the Matrix pth Root and its Inverse. SIAM J. Matrix Anal. Appl. 2006.
- ...

N.J. Higham, Computing the polar decomposition - with applications, *SIAM J. Sci. Stat. Comput.* 7 (1986).

Polar decomposition

$$A = UH$$

$$A \in \mathbb{C}^{n \times n}, \quad \text{nonsingular}$$

U - unitary, H - Hermitian positive definite

Iterative Algorithms for $A = UH$

$$X_0 = A, \qquad \lim_{k\to\infty} X_k = U$$

$$H = \frac{1}{2}(U^H A + A^H U)$$

Higham's method

$$X_{k+1} = \frac{1}{2}\left(\gamma_k X_k + \frac{1}{\gamma_k} X_k^{-H}\right), \qquad X_0 = A$$

γ_k – scaling parameters

Scaling parameters

- 1986, Higham: optimal and $(1, \infty)$
- 1999, Kiełbasiński: quasi-optimal
  presented by me: ILAS (Barcelona) and Householder Symposium (Whistler)
- 2008, Byers and Xu: quasi-optimal

Analysis of numerical behaviour of Higham's method for polar decomposition - very long story

Very long Polish story

- **1993**, my student Paweł Zieliński, Master thesis - numerical experiments with algorithms for polar decomposition
- **1993**, our review paper on polar decopomsition - submitted to Applied Mathematics, Annals of the Polish Mathematical Society
- **1993**, A. Kiełbasiński (a referee) has started the error analysis of Higham's method
- **1995**, the review paper published
- **2003**, the first (theoretical) part of numerical analysis of Higham's method has been published - the second (experimental) never has appeared(only reports). Kiełbasiński was the leader of this research.
- **2015**, short version on numerical experiments published

Andrzej Kiełbasiński (1927- 2019), University of Warsaw

Polish translation of J.H. Wilkinson "Rounding errors in algerbraic process" (in 1967)

Now Paweł Zieliński prof. Wrocław University of Technology

Question of Kiełbasiński:

- Which model of matrix inversion is neccesary to have good behaviour of Higham's method?

Numerical correctness of matrix inversion (NC property)
G computed inverse

$$\exists \Delta_X, \Delta_G : \quad G = (X + \Delta_X)^{-1} + \Delta_G, \tag{1}$$

where

$$\|\Delta_X\|_F \leqslant \varepsilon_X \|X\|_2, \qquad \|\Delta_G\|_F \leqslant \varepsilon_G \|G\|_2. \tag{2}$$

Artificial example of inversion for numerical experiments

$$X = \mathrm{diag}(c, \sqrt{c}, 1), \qquad G = X^{-1} + \Gamma, \qquad c > 1$$

- In the scaled Higham's method matrix inversion should yield the computed inverse G of the matrix X (the inverse of the current iterate) having the NC property (numerical correctness). This property is warranted by the inversion *via* GECP.
- Using the standard inversion *via* GEPP can fail, yielding for some special matrices A a poor unitary polar factor $\widetilde{U}$. This will never occur for well-conditioned matrices A.
- Corollaries from numerical experiments were consistent with the analysis of Nakatsukasa and Higham 2012.

1. **1995** P. Zieliński, K. Ziętak, Polar decomposition - properties, applications and algorithms, Applied Math.
2. **2003** Kiełbasiński, K. Ziętak, Numerical behaviour of Higham's scaled method for polar decomposition, Numer. Algorithms
3. **2008** R. Byers, H. Xu, A new scaling for Newton's iteration for the polar decomposition and its backward stability, SIMAX
4. **2010** A. Kiełbasiński, K. Ziętak, Note on "A new scaling for Newton's iteration for the polar decomposition and its backward stability" by R. Byers and H. Xu, SIMAX
5. **2012** Y. Nakatsukasa, N.J. Higham, Backward stability of iterations for computing the polar decomposition, SIMAX
6. **2015** A. Kiełbasiński, P. Zieliński, K. Ziętak, On iterative algorithms for the polar decomposition of a matrix and the matrix sign function, Applied Mathematics and Computation

Matrix functions were an unfamiliar topic for me until researching of Nick. I was happy to work in this very interesting field.

I would like to mention only one examples of influences of Nick: rational matrix iterations preserving structure

rational matrix iteration, $h(z)$ rational function

$$X_0 = A, \qquad X_{k+1} = h(X_k)$$

Rational matrix iteration preserves structure of group $\mathcal{G}$ if $A \in \mathcal{G}$ implies all iterates lie in $\mathcal{G}$

Higham, Mackey, Mackey, Tisseur

- N.J. Higham, D.S. Mackey,N. Mackey, F. Tisseur, Computing the Polar Decomposition and the Matrix Sign Decomposition in Matrix Groups. SIAM J. Matrix Anal. Appl. 2004.
- N.J. Higham, D.S. Mackey,N. Mackey, F. Tisseur, Functions Preserving Matrix Groups and Iterations for the Matrix Square Root. SIAM J. Matrix Anal. Appl. 2005.

Rational matrix iteration preserves structure of group of automorphism associated with bilinear or sesquilinear form if

$$h(z) = \pm z^{\ell} \frac{w(z)}{rev\ w(z)}$$

$w(z)$ polynomial, $rev(z)$ coefficients reversed

Beata Laszkiewicz and KZ prove that principal Pade iteration for the matrix sector function preserves structure in group of automorphism.
It was an extension of the result for the sign matrix function, proven by Nick and coauthors.

Definition

Let eigenvalues λ_j of nonsingular A satisfy:

$$\arg(\lambda_j) \neq \frac{2q\pi}{p} + \frac{\pi}{p} \quad \text{for } q = 0, \ldots, p-1$$

Then

$$\mathrm{sect}_p(A) = A(A^p)^{-1/p}$$

B. Laszkiewicz, KZ, A Pade family of iterations for the matrix sector function and the matrix pth root, Numerical Linear Algebra with Applications 16 (2009)

Very technical proof! Zeilberger algorithm applied

Let

$$F(j) = \frac{(m-j)!}{m!} \sum_{\ell=j}^{m} \frac{(-1)^{\ell+j}(2m-\ell)!p^{m-\ell}}{(m-\ell)!(\ell-j)!} \prod_{t=m-\ell+1}^{m-j} (tp+1)$$

We show that

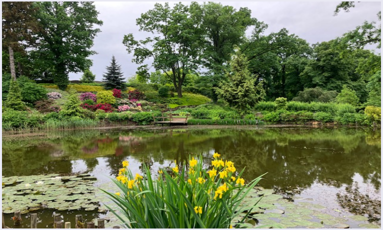
$$F(j) = \prod_{\ell=j+1}^{m} (\ell p - 1)$$

Dear Nick,

I would like to thank you for always being there for me, like a very young brother.

Your excellent book "Functions of matrices. Theory and computation" is my favourite one.

Krystyna



A view from Poland with greetings